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# Analytical solutions in the problem of the optimal control of the rotation of an axisymmetric body☆

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**Abstract**

The problem of the optimal control of the rotation of an axisymmetric rigid body is investigated. An integral functional, characterizing the power consumption to carry out a manoeuvre is chosen as the criterion, and the boundary conditions for the angular velocity vector are arbitrary. The principal moment of the applied external forces serves as the control. The necessary conditions of the maximum principle are used to solve the problem in the case of a fixed completion time. New non-trivial first integrals are established for the canonical system of direct and conjugate differential equations obtained, which enable the set of all extremals to be parametrized. Hence, the optimal-control problem is reduced to a problem of non-linear mathematical programming. It is shown that there cannot be more than two different solutions in the latter, and a family of boundary conditions is established when the optimum rotation is determined in a uniquely explicit form.


Many similar control problems have already been investigated by other researchers for more general assumptions regarding the system being controlled (an asymmetrical inertia tensor, geometrical restrictions on the control vector, a time-optimality criterion, etc.). However, it is assumed that the final condition for the velocity vector or the angular momentum possesses the property of symmetry (in particular, complete stop of the rotations). For example, the problem of the optimal control of the rotation of an axisymmetric rigid body is described in detail in Refs. 1–3; in particular, the effective use of the property of a system with an invariant norm for the analytical solution of the problem of the fastest braking of a rotating body with dynamic symmetry was demonstrated. However, in view of the considerable non-linearity of the problem, constructive results for arbitrary boundary conditions and other quality functionals were not obtained. In this paper, by selecting a family of first integrals for the direct and conjugate system of equations of the maximum principle, we are able to construct a number of analytical solutions.

## 1. Formulation of the problem

We will investigate the problem of the optimal control of the value and direction of the rotation velocity of an axisymmetric rigid body. We use as the control the principal moment of the external forces applied to the body. The basic control problem is taken to be the change in the angular velocity vector from the initial value to the required terminal value in a fixed finite time in such a way that a manoeuvre requires the least power consumption. The boundary

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conditions for the angular velocity vector can be arbitrary, while any change in orientation is ignored. In other words, the control problem of spin-up/spin-down is investigated.

We will assume that the axes of the connected system of coordinates coincide with the principal central axes of inertia of the body. Suppose $I_i > 0$ ($i = 1, 2, 3$) are the principal central moments of inertia by virtue of the axial symmetry of the body

$$I_1 = I_2 = I \tag{1.1}$$

The corresponding Euler dynamic equations in projections onto the coupled axes coordinate-by-coordinate can be written in the form

$$\dot{\omega}_j = (-1)^j k\omega_{3-j}\omega_3 + u_j, \quad j = 1, 2; \quad \dot{\omega}_3 = u_3; \quad k = (I_2 - I_3)/I_1 = 1 - I_3/I \tag{1.2}$$

where $\omega = (\omega_1, \omega_2, \omega_3)^T$ is the angular velocity vector and $u = (u_1, u_2, u_3)^T$ is the control vector, which is related to the vector of the principal moment of the external forces $M = (M_1, M_2, M_3)^T$ as follows:

$$u_i = I_i^{-1} M_i, \quad i = 1, 2, 3$$

For the spin-up/spin-down manoeuvre being investigated, we will assume the following specified boundary conditions

$$\omega(0) = \upsilon, \quad \omega(T) = w; \quad \upsilon = (\upsilon_1, \upsilon_2, \upsilon_3)^T, \quad w = (w_1, w_2, w_3)^T \tag{1.3}$$

where $T > 0$ is the instant when the motion is completed, which is also assumed known.

We will choose the following integral functional as the criterion characterizing the value of the total power consumption to carry out the required manoeuvre (everywhere henceforth integration is carried out over the interval $[0,T]$)

$$J = \frac{1}{2}\int (u_1^2(t) + u_2^2(t) + C^{-1}u_3^2(t))dt \tag{1.4}$$

where $C > 0$ is a weighting factor. In particular if $C = I^2 I_3^{-2}$, we have

$$J = \frac{1}{2} I^{-2} \int \langle M(t), M(t) \rangle dt$$

where $\langle .,. \rangle$ is the symbol of a scalar product. The purpose of the control is to minimize the power consumption

$$\inf J$$

where the lower bound is sought over all the admissible trajectories and controls (in the sense of boundary conditions (1.3)).

Hence, the problem of the optimal control of system (1.2), (1.3) with cost function (1.4) is non-linear, which makes it extremely difficult to obtain an accurate analytical solution. The purpose of this paper is to describe new first integrals of the canonical system of the maximum principle and to obtain analytical solutions for control.

## 2. The use of the maximum-principle formalism

The optimal control problem will be investigated using the necessary conditions of the maximum principle. The main assumption which is used in the following discussion will be the assumption that a solution of the problem exists in the class of piecewise-continuous controls. It can be shown that extension of the class of controls to measurable time functions does not lead to any improvement in the result.

The variational problem being investigated is not degenerate, and hence the conjugate variable corresponding to the cost functional $J$, in view of the homogeneity of the equations, is assumed to be equal to $-1$. Hence we put

$$H(\omega, \gamma, u) = -1/2(u_1^2 + u_2^2 + C^{-1}u_3^2) + k\gamma_1\omega_2\omega_3 - k\gamma_2\omega_1\omega_3 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3 \tag{2.1}$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)^T$ is the vector of the conjugate variables.

From the condition for a maximum of the function (2.1) with respect to the control vector we obtain

$$u_1 = \gamma_1, \quad u_2 = \gamma_2, \quad u_3 = C\gamma_3 \tag{2.2}$$

After this, we can write the following direct and conjugate systems

$$\begin{aligned} &\dot{\omega}_j = (-1)^j k\omega_{3-j}\omega_3 + \gamma_j, \quad \dot{\gamma}_j = (-1)^j k\gamma_{3-j}\omega_3, \quad j = 1, 2; \quad \dot{\omega}_3 = C\gamma_3, \\ &\dot{\gamma}_3 = (\gamma_2\omega_1 - \gamma_1\omega_2)k \end{aligned} \tag{2.3}$$

Substituting expressions (2.2) into Eq. (2.1) we obtain the following expression for the Hamilton function

$$H(\omega, \gamma) = (\gamma_1\omega_2 - \gamma_2\omega_1)k\omega_3 + 1/2(\gamma_1^2 + \gamma_2^2 + C\gamma_3^2) \tag{2.4}$$

As is well-known, the direct and conjugate systems (2.3), corresponding to optimal control (2.2), form a system of canonical differential equations with Hamilton function $H(\omega, \gamma)$, defined by formula (2.4), i.e.

$$\dot{\omega} = \partial H(\omega, \gamma)/\partial\gamma, \quad \dot{\gamma} = -\partial H(\omega, \gamma)/\partial\omega \tag{2.5}$$

Moreover, we can conclude from Eqs. (2.3) and (2.2) that the optimal control and, consequently, the trajectory belong to the class of infinitely differential functions of time.

The Hamilton system obtained by using the formalism of the maximum principle has the obvious first integral

$$H(\omega, \gamma) = h_1 = \text{const} \tag{2.6}$$

which reflects the property that the Hamilton function is constant for a system of canonical equations which are explicitly independent of time.

We will show later that there are other first integrals. This enables us to integrate system (2.3) completely and to write explicit formulae for the extremals.

## 3. First integrals

For system (2.3) we can write a family of first integrals which are a consequence of the properties of the defined symmetry of the Hamilton function. More accurately, the following equalities hold for the solutions of system (2.3)

$$\gamma_1\omega_2 - \gamma_2\omega_1 = h_2 = \text{const}, \quad \gamma_1^2 + \gamma_2^2 = h_3 = \text{const} \tag{3.1}$$

This fact can easily be verified by direct differentiation. We will show, nevertheless, how these relations are connected with the existing single-parameter groups of coordinate transformations, which leave the Hamilton function unchanged. Note, incidentally, that the first integral (2.6) is a consequence of the invariance of the Hamilton function under groups of time shifts.

We will denote the matrix of rotation by an angle $\varepsilon$ in the plane as follows:

$$B(\varepsilon) = \begin{Vmatrix} \cos\varepsilon & \sin\varepsilon \\ -\sin\varepsilon & \cos\varepsilon \end{Vmatrix} \in SO(2), \quad \varepsilon \in \mathbb{R} \tag{3.2}$$

and we will introduce the following notation for the vectors and matrices

$$\begin{aligned} &\omega^\varepsilon = (\omega_1^\varepsilon, \omega_2^\varepsilon, \omega_3^\varepsilon)^T = \begin{Vmatrix} B(\varepsilon) & 0 \\ 0 & 1 \end{Vmatrix}\omega, \quad \gamma^\varepsilon = (\gamma_1^\varepsilon, \gamma_2^\varepsilon, \gamma_3^\varepsilon)^T = \begin{Vmatrix} B(\varepsilon) & 0 \\ 0 & 1 \end{Vmatrix}\gamma \\ &\tilde{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \ \tilde{\gamma} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \ \tilde{\omega}^\varepsilon = \begin{pmatrix} \omega_1^\varepsilon \\ \omega_2^\varepsilon \end{pmatrix}, \ \tilde{\gamma}^\varepsilon = \begin{pmatrix} \gamma_1^\varepsilon \\ \gamma_2^\varepsilon \end{pmatrix}, \ \tilde{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \ \tilde{\upsilon} = \begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix}, \ \tilde{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \end{aligned} \tag{3.3}$$

$$\Theta = \left\|\begin{matrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}\right\|, \quad \tilde{\Theta} = \left\|\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}\right\|, \quad D(\tilde{\omega}, \tilde{\gamma}) = \det\left\|\begin{matrix} \omega_1 & \gamma_1 \\ \omega_2 & \gamma_2 \end{matrix}\right\|$$

The following equalities hold

$$D(\tilde{\gamma}^{\varepsilon}, \tilde{\omega}^{\varepsilon}) = D(B(\varepsilon)\tilde{\gamma}, B(\varepsilon)\tilde{\omega}) = \det B(\varepsilon) D(\tilde{\gamma}, \tilde{\omega}) = D(\tilde{\gamma}, \tilde{\omega})$$

$$(\gamma_1^{\varepsilon})^2 + (\gamma_2^{\varepsilon})^2 = \gamma_1^2 + \gamma_2^2, \quad \omega_3^{\varepsilon} = \omega_3, \quad \gamma_3^{\varepsilon} = \gamma_3$$

This, in turn, means that for any real $\varepsilon$ the following relation holds

$$F(\tilde{\gamma}^{\varepsilon}, \tilde{\omega}^{\varepsilon})k\omega_3^{\varepsilon} + \frac{1}{2}((\gamma_1^{\varepsilon})^2 + (\gamma_2^{\varepsilon})^2 + C(\gamma_3^{\varepsilon})^2) = D(\tilde{\gamma}, \tilde{\omega})k\omega_3 + \frac{1}{2}(\gamma_1^2 + \gamma_2^2 + C\gamma_3^2)$$

or, if we take formula (2.4) into account,

$$H(\varepsilon) \equiv H(\omega^{\varepsilon}, \gamma^{\varepsilon}) = H(\omega, \gamma) \equiv H(0) \tag{3.4}$$

In other words, the Hamilton function is invariant under a single-parameter group of linear transformations, described by formulae (3.4).

Differentiating both sides of Eq. (3.4) with respect to $\varepsilon$, we obtain

$$\left.\frac{d}{d\varepsilon}H(\varepsilon)\right|_{\varepsilon = 0} = 0 \tag{3.5}$$

or, in more detail, bearing in mind canonical Eq. (2.5),

$$\left(\frac{\partial H(\omega^{\varepsilon}, \gamma^{\varepsilon})}{\partial \omega^{\varepsilon}}\right)^T \left.\frac{d\omega^{\varepsilon}}{d\varepsilon}\right|_{\varepsilon = 0} + \left(\frac{\partial H(\omega^{\varepsilon}, \gamma^{\varepsilon})}{\partial \gamma^{\varepsilon}}\right)^T \left.\frac{d\gamma^{\varepsilon}}{d\varepsilon}\right|_{\varepsilon = 0} = 0 \tag{3.6}$$

It follows from (3.3) that

$$\left.\frac{d\omega^{\varepsilon}}{d\varepsilon}\right|_{\varepsilon = 0} = \Theta\omega, \quad \left.\frac{d\gamma^{\varepsilon}}{d\varepsilon}\right|_{\varepsilon = 0} = \Theta\gamma$$

We then obtain from (3.3) and (2.5)

$$-\dot{\gamma}^T\Theta\omega + \dot{\omega}^T\Theta\gamma = 0$$

or, alternatively,

$$-\dot{\tilde{\gamma}}^T\tilde{\Theta}\tilde{\omega} + \dot{\tilde{\omega}}^T\tilde{\Theta}\tilde{\gamma} = 0$$

Using the fact that the matrix $\tilde{\Theta}$ is skew-symmetric, we arrive at the equality

$$\frac{d}{dt}(\tilde{\omega}^T\tilde{\Theta}\tilde{\gamma}) = \frac{d}{dt}D(\tilde{\gamma}, \tilde{\omega}) = 0$$

whence relation (3.1) follows.

To derive the second formula of (3.1) we will use another group of linear transformations, without changing the Hamilton function. For real $\varepsilon$ we put

$$\omega_j^{\varepsilon} = \omega_j + \varepsilon\gamma_j, \quad \gamma_j^{\varepsilon} = \gamma_j, \quad j = 1, 2; \quad \omega_3^{\varepsilon} = \omega_3, \quad \gamma_3^{\varepsilon} = \gamma_3$$

Equality (3.4) follows from expression (2.4) and the elementary properties of a determinant, and consequently equality (3.5) holds.

Taking the following relations into account in formula (3.6)

$$\left.\frac{d\omega^{\varepsilon}}{d\varepsilon}\right|_{\varepsilon=0} = \operatorname{diag}\{1, 1, 0\}\gamma, \quad \left.\frac{d\gamma^{\varepsilon}}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

we arrive at the equation

$$\dot{\gamma}^T \operatorname{diag}\{1, 1, 0\}\gamma = 0$$

whence the integral, defined by the second equation of (3.1), follows.

Hence, in the problem considered the presence in the Hamilton function of two groups of invariant transformations (plane-rotation and linear-shift groups) has led to the appearance of two new first integrals. This enables us to solve the initial optimal-control problem analytically.

## 4. The family of extremals

We will now integrate Eq. (2.3), the solutions of which are extremals in the initial problem.

In view of the first equality, the equations of the direct and conjugate systems for the third coordinates are independent of the remaining ones, which enables them to be investigated separately. Taking the first integral, defined by the first equality of (3.1), into account, we obtain

$$\dot{\omega}_3 = C\gamma_3, \quad \dot{\gamma}_3 = -h_2 k$$

and we can therefore write the corresponding solutions

$$\gamma_3(t) = a_1 t + a_2, \quad \omega_3(t) = \frac{Ca_1 t^2}{2} + Ca_2 t + a_3$$

where we have used the notation

$$a_1 = -h_2 k \tag{4.1}$$

and $a_2$ and $a_3$ are certain constants.

We will write the remaining equations of system (2.3) in matrix form

$$\dot{\tilde{\omega}} = k\omega_3\tilde{\Theta}\tilde{\omega} + \tilde{\gamma}, \quad \dot{\tilde{\gamma}} = k\omega_3\tilde{\Theta}\tilde{\gamma} \tag{4.2}$$

Suppose $B(t) \in SO(2)$ is the matrix of rotation in a plane, which is the solution of the Cauchy problem

$$B(0) = \operatorname{diag}\{1, 1\}, \quad \dot{B} = k\omega_3\tilde{\Theta}B; \tag{4.3}$$

As is well-known, the following parametrization holds

$$B \equiv B(t) = B(\varphi(t)) \tag{4.4}$$

where $\varphi(t)$ is the angle of rotation, which satisfies the equations

$$\varphi(0) = 0(\operatorname{mod}2\pi), \quad \dot{\varphi} = k\omega_3 \tag{4.5}$$

which follows directly from expressions (4.3) and (4.4).

Bearing in mind that the function $\omega_3(t)$ has already been determined, the solution of the Cauchy problem (4.5) can be written as

$$\varphi(t) = \frac{Cka_1 t^3}{6} + \frac{Cka_2 t^2}{2} + ka_3 t(\operatorname{mod}2\pi) \tag{4.6}$$

From Eqs. (4.2) and (4.3) we obtain the relations

$$(B^T\tilde{\omega})^{\cdot} = B^T\tilde{\gamma}, \quad (B^T\tilde{\omega})^{\cdot} = 0$$

whence

$$\tilde{\omega}(t) = B(\varphi(t))(\tilde{f}t+\tilde{g}), \quad \tilde{\gamma}(t) = B(\varphi(t))\tilde{f}, \quad \tilde{f} = (f_1, f_2)^T, \quad \tilde{g} = (g_1, g_2)^T$$

where $\tilde{f}$, $\tilde{g}$ are constant vectors.

Hence, the general solution of system (2.3) can be written in the form

$$\begin{aligned} &\tilde{\omega}(t) = B(\varphi(t))(\tilde{f}t+\tilde{g}), \quad \omega_3(t) = \frac{Ca_1t^2}{2}+Ca_2t+a_3 \\ &\tilde{\gamma}(t) = B(\varphi(t))\tilde{f}, \quad \gamma_3(t) = a_1t+a_2 \end{aligned} \tag{4.7}$$

The angle $\varphi(t)$ is described by Eq. (4.6).

We will now obtain the solution of the two-point boundary-value problem (1.3), (2.3). Using conditions (1.3) with $t=0$ we can determine some of the constants

$$g_1 = \upsilon_1, \quad g_2 = \upsilon_2, \quad a_3 = \upsilon_3 \tag{4.8}$$

It is possible to reduce the number of unknown parameters by using equality (4.1) and the first integrals. In fact, from the first formula of (3.1) and Eqs. (4.7) we obtain the relations

$$h_2 = D(\tilde{\gamma}, \tilde{\omega}) = D(B\tilde{f}, B(\tilde{f}t+\tilde{g})) = D(B(\tilde{f}, \tilde{f}t+\tilde{g})) = (\det B)D(\tilde{f}, \tilde{f}t+\tilde{g}) = F(\tilde{f}, \tilde{g})$$

Now, using formula (4.1), we arrive at the equality

$$a_1 = -kD(\tilde{f}, \tilde{\upsilon}) \tag{4.9}$$

which, in fact, denotes that the constant $a_1$ depends on the unknown vector of the parameters $\tilde{f}$.

Hence, only the three constants $f_1$, $f_2$ and $a_2$ remain to be determined. To do this we use the second boundary condition (1.3) with $t=T$. Transformations lead to a system of three equations in three unknowns

$$\tilde{f} = [B^T(\varphi(t))\tilde{w}-\tilde{\upsilon}]T^{-1}, \quad -\frac{kT}{2}D(\tilde{f}, \tilde{\upsilon})+a_2 = \frac{w_3-\upsilon_3}{CT} \tag{4.10}$$

where

$$\varphi(T) = -\frac{Ck^2T^3}{6}D(\tilde{f}, \tilde{\upsilon})+\frac{Cka_2T^2}{2}+k\upsilon_3T$$

It is not possible to obtain a general solution of this problem due to the considerable non-linearity. However, we can make some observations on the properties of these solutions: (1) the solution may not be unique, but always exists (which follows, for example, from mechanical considerations on the nature of the problem being solved), (2) for each solution there is a corresponding trajectory in the initial problem, which satisfies the necessary conditions of the maximum principle, and vice versa. Consequently, we obtain an exhaustive description of a family of extremal trajectories and the corresponding equations in the problem of the optimal control being investigated. We will further consider the problem of finding a minimum of the initial functional in the set of extremals obtained.

## 5. Solution of the problem of the optimal control of rotation

Here we will consider the choice of the optimal trajectory from the family of extremal trajectories obtained. We will therefore investigate the problem of finding the minimum of the functional on the parametrized set of extremals and, hence, we will change from a variational problem in functional space to a problem of mathematical programming in a finite-dimensional space. For convenience we will introduce another parametrization of the extremal trajectories and equations, which enables us to use the structure of the initial minimized functional.

Substituting expressions (2.2) into (1.4), we obtain

$$2J = \int(\gamma_1^2(t)+\gamma_2^2(t))dt+C\int\gamma_3^2(t)dt \tag{5.1}$$

We will convert the first term. Using the second equation of (3.1), we have

$$\int(\gamma_1^2(t)+\gamma_2^2(t))dt = \langle\tilde{\gamma}(0), \tilde{\gamma}(0)\rangle T$$

It follows from the third formula of (4.7) that $\tilde{\gamma}(0) = \tilde{f}$. Hence, using the first equality and the properties of the orthogonal matrix $B(\varphi(t))$, we have

$$\langle\tilde{\gamma}(0), \tilde{\gamma}(0)\rangle = \langle\tilde{f}, \tilde{f}\rangle = \{\langle B^T(\varphi(T))\tilde{w}, B^T(\varphi(T))\tilde{w}\rangle - 2\langle B^T(\varphi(T))\tilde{w}, \tilde{\upsilon}\rangle + \langle\tilde{\upsilon}, \tilde{\upsilon}\rangle\}T^{-2} =$$
$$= \{\langle\tilde{w}, \tilde{w}\rangle + \langle\tilde{\upsilon}, \tilde{\upsilon}\rangle\}T^{-2} - 2\langle B^T(\varphi(T))\tilde{w}, \tilde{\upsilon}\rangle T^{-2}$$

We will assume both vectors $\tilde{\upsilon}$ and $\tilde{w}$ to be non-zero. Then

$$B(\alpha) = \frac{\tilde{\upsilon}}{|\tilde{\upsilon}|} = \frac{\tilde{w}}{|\tilde{w}|}$$

where $\alpha$ is the angle between these vectors. To simplify the calculations we will also assume that if $|\tilde{\upsilon}||\tilde{w}| = 0$ then $\alpha = 0$. From the definition of the scalar product of vectors we can write

$$\langle B^T(\varphi(T))\tilde{w}, \tilde{\upsilon}\rangle = |\tilde{w}||\tilde{\upsilon}|\cos B(\varphi(T))\tilde{\upsilon}, \quad \tilde{w} = |\tilde{\upsilon}||\tilde{w}|\cos(\varphi(T)-\alpha)$$

Hence,

$$\int(\gamma_1^2(t)+\gamma_2^2(t))dt = (|\tilde{\upsilon}|^2+|\tilde{w}|^2)T^{-1} - 2|\tilde{\upsilon}||\tilde{w}|\cos(\varphi(T)-\alpha)T^{-1}$$

If we use the third equality of (4.7), the second term in formula (5.1) can be written in the form

$$C\int\gamma_3^2(t)dt = C\int(a_1t+a_2)^2dt = C\left(\frac{a_1^2T^3}{3}+a_1a_2T^2+a_2^2T\right) \tag{5.2}$$

Consequently, taking into account equality (4.6), which describes the value of the angle $\varphi(t)$, for the objective functional, defined on the family of extremals, we have

$$\begin{aligned} 2J &= (|\tilde{\upsilon}|^2+|\tilde{w}|^2)T^{-1} - 2|\tilde{\upsilon}||\tilde{w}|\cos\left(\frac{Cka_1T^3}{6}+\frac{Cka_2T^2}{2}+k\upsilon_3T-\alpha\right)T^{-1} + \\ &+ C\left(\frac{a_1^2T^3}{3}+a_1a_2T^2+a_2^2T\right) \end{aligned} \tag{5.3}$$

The boundary conditions at the left end for all the coordinates and the conditions at the right end for the first two components of the angular velocity vector have been taken into account by means of relations (4.8) and the first equality of (4.10). From boundary condition (1.3) at the final instant for the remaining component of the angular velocity, since by construction $CT > 0$, we can express $a_2$ as follows:

$$a_2 = \frac{w_3-\upsilon_3}{CT} - \frac{a_1T}{2} \tag{5.4}$$

and we substitute it into Eq. (5.3). As a result of these transformations we arrive at the conclusion that the problem of finding the minimum value of the functional on a set of extremals reduces to the problem of minimizing a function of one variable without restrictions

$$\inf_{a_1} J(a_1); \quad 2J(a_1) = (|\tilde{\upsilon}|^2+|\tilde{w}|^2)T^{-1} - 2|\tilde{\upsilon}||\tilde{w}|\cos(x+b)T^{-1} + \frac{Ca_1^2T^3}{12} + \frac{(w_3-\upsilon_3)^T}{CT}$$
$$x = -\frac{kCT^3}{12}a_1, \quad b = \frac{w_3+\upsilon_3}{2}kT - \alpha(\mathrm{mod}2\pi)$$

The problem in question is then equivalent to the problem of finding the exact lower bound of the function $F(x)$

$$\inf_x F(x); \quad F(x) = -2a\cos(x+b) + x^2; \quad a = \frac{Ck^2T^2}{12}|\tilde{\upsilon}||\tilde{w}| \geq 0 \tag{5.5}$$

We recall that the boundary conditions and the completion time here are assumed to be fixed.

By a corollary of Weierstrass' theorem [Ref. 4, Chapter 1, Section 2, Paragraph 2.3] an exact lower bound can be attained and consequently a solution of the problem exists

$$F(x^*) = \min_x F(x)$$

If

$$|\tilde{\upsilon}||\tilde{w}| = 0 \tag{5.6}$$

then $a=0$, and we arrive at the problem

$$F(x^*) = \min_x x^2$$

which has the unique solution

$$x^* = 0, \quad F(x^*) = 0$$

Hence, if $|\tilde{\upsilon}||\tilde{w}| = 0$, we have

$$a_1 = 0, \quad a_2 = \frac{w_3 - \upsilon_3}{CT}$$

Using formulae (4.6) and (4.7), we now obtain the corresponding solutions for some of the variables

$$\varphi(t) = \frac{w_3 - \upsilon_3}{CT}\frac{kt^2}{2} + k\upsilon_3 t (\mathrm{mod} 2\pi), \ u_3(t) = C\gamma_3(t) = \frac{w_3 - \upsilon_3}{T}, \ \omega_3(t) = \frac{(w_3 - \upsilon_3)}{T}t + \upsilon_3 \tag{5.7}$$

We will write the solutions for the remaining two coordinates of the angular velocity vector and the control by distinguishing two special cases, for which equality (5.6) is possible

$$1)\ \tilde{\upsilon} = 0 \Rightarrow \tilde{f} = B^T(\varphi(T))\tilde{w}T^{-1}, \quad \tilde{u}(t) = \tilde{\gamma}(t) = \tilde{\omega}(t) = B(\varphi(t) - \varphi(T))\tilde{w}T^{-1}$$

$$2)\ \tilde{w} = 0 \Rightarrow \tilde{f} = -\tilde{\upsilon}T^{-1}, \quad \tilde{u}(t) = \tilde{\gamma}(t) = -B(\varphi(t))\tilde{\upsilon}T^{-1}, \quad \tilde{\omega}(t) = B(\varphi(t))\tilde{\upsilon}(1 - T^{-1})$$

We will now investigate the solution of the problem assuming that $|\tilde{\upsilon}||\tilde{w}| \neq 0$, i.e. $a>0$. Here it will be of interest to present the most constructive and simple description of the solution of the corresponding general problem of non-linear mathematical programming (5.5).

We will assume that $A=\{x_1, \ldots, x_n\}$ is a set of all local minima of the function $F(x)$. In other words,

$$A = \{x \in \mathbb{R}: F'(x) = 0, F''(x) \geq 0\}$$

or, in more detail,

$$\sin(x+b) = -a^{-1}x, \quad \cos(x+b) \geq -a^{-1}, \quad x \in A \tag{5.8}$$

The finiteness of the number of elements in the set $A$ follows from the equality.

It is obvious that the following relation holds

$$\min_x F(x) = \min_{x \in A} F(x)$$

For $x \in A$, by virtue of the definition, another description of the minimized function is possible, namely,

$$F(x) = -2a\cos(x+b) + a^2\sin^2(x+b) = a^2 - G(a\cos(x+b))$$

$$G(y) = 2y + y^2$$

Since

$$\min_{x \in A} F(x) = a^2 - \max_{x \in A} G(a\cos(x+b))$$

we arrive at the problem

$$\max_{x \in A} G(a\cos(x+b))$$

The function $G(y)$ has a unique global minimum at the point $y^* = -1$. Consequently, the maximum of the function $G(a\cos(x+b))$ in $x \in A$ is reached at points for which the values of $a\cos(x+b)$ are the greatest distance from $y^*$. By virtue of the second condition of (5.8) this means that this problem is equivalent to a problem of the form

$$\max_{x \in A} \cos(x+b)$$

Suppose

$$A_1 = \{x \in A\colon \cos(x+b) \ge 0\}, \quad A_2 = \{x \in A\colon \cos(x+b) < 0\}$$

We will arrange the elements of these sets in order of increasing absolute values

$$A_1 = \{\bar{x}_1, \ldots, \bar{x}_k\}, \quad |X_*| \equiv |\bar{x}_1| < |\bar{x}_2| < \ldots < |\bar{x}_k| \le a$$

$$A_2 = \{\tilde{x}_1, \ldots, \tilde{x}_m\}, \quad |\tilde{x}_1| < |\tilde{x}_2| < \ldots < |\tilde{x}_m| \equiv |X^*| \le a$$

By construction

$$\cos(x+b) = \begin{cases} \sqrt{1-a^{-2}x^2}, & x \in A_1 \\ -\sqrt{1-a^{-2}x^2}, & x \in A_2 \end{cases}$$

and hence

$$\max_{x \in A_1} \cos(x+b) = \cos(X_* + b), \quad \max_{x \in A_2} \cos(x+b) = \cos(X^* + b)$$

Since $A = A_1 \cup A_2$, we have

$$\max_{x \in A} \cos(x+b) = \max\{\cos(X_* + b), \cos(X^* + b)\}$$

and consequently,

$$x^* \in \{X_*, X^*\}$$

$$F(x^*) = a^2 - 2a\max\{\cos(X_* + b), \cos(X^* + b)\} - a^2\max{}^2\{\cos(X_* + b), \cos(X^* + b)\}$$

Because of the considerable non-linearity of the problem in question, it is hardly possible to obtain a more constructive description of the solution. Nevertheless, some qualitative conclusions can be drawn.

In fact, if $a > 0$, the solution of problem (5.5) may turn out not to be unique, but there cannot be more than two solutions, since $x^* \in \{X_*, X^*\}$. The optimal trajectory and the corresponding control are described by Eqs. (4.6)–(4.10) and (5.4), in which we must put

$$a_1 = -\frac{12}{kCT^3}x^*$$

In the general situation, $a > 0$ is nevertheless a special case, when it is possible to obtain an explicit description of the optimal trajectory and prove the uniqueness of the solution.

In fact, suppose

$$b = 0$$

or, in more detail,

$$\frac{w_3 + \upsilon_3}{2}kT - \alpha = 0(\mathrm{mod}2\pi) \tag{5.9}$$

The solution of problem (5.5) is then unique and has the form

$$x^* = 0, \quad F(x^*) = -2a$$

by virtue of simple upper estimates

$$\min_x\{-2a\cos x + x^2\} \ge -2a\max_x \cos x + \min_x x^2 = -2a$$

Then

$$a_1 = 0, \quad a_2 = \frac{w_3 - \upsilon_2}{CT} \tag{5.10}$$

and consequently, $\varphi(t)$, $u_3(t)$ and $\omega_3(t)$ are described by relations (5.7). From condition (5.9) and the first equality of (4.10) we then obtain

$$\varphi(T) = \alpha(\mathrm{mod}2\pi), \quad \tilde{f} = (B^T(\varphi(T))\tilde{w} - \tilde{\upsilon})T^{-1} = (B^T(-\alpha)\tilde{w} - \tilde{\upsilon})T^{-1}$$

Using these relations in the general solution (4.7), we obtain

$$\tilde{u}(t) = \tilde{\gamma}(t) = B(\varphi(t) - \alpha)\tilde{w}T^{-1} - B(\varphi(t))\tilde{\upsilon}T^{-1}$$

$$\tilde{\omega}(t) = B(\varphi(t) - \alpha)\tilde{w}T^{-1} + B(\varphi(t))\tilde{\upsilon}(1 - T^{-1}t)$$

Equality (5.9) holds, for example, for the boundary conditions

$$\tilde{\upsilon} = \tilde{w}, \quad \upsilon_3 = -w_3$$

In this case

$$\alpha = 0(\mathrm{mod}2\pi); \quad \tilde{u}(t) \equiv 0, \quad \tilde{\omega}(t) \equiv B(\varphi(t))\tilde{\upsilon}$$

This, in particular, indicates that, for general boundary conditions, the optimal rotation, generally speaking, does not imply complete stop at any instant. In other words, a manoeuvre consisting of two stages (deceleration to complete stop and subsequent spin-up), is not, in general, optimal from the point of view of minimum power consumption.

In conclusion, we will show that the well-known solution of the problem of the optimal control of the rotation of a spherically symmetrical body can be obtained as a special case of the discussions presented above. If $I_1 = I_2 = I_3$, then $k = 0$, and we arrive at the conclusion that the problem of searching for a minimum of the functional on extremals reduces to the following

$$\min_{a_1} a_1^2$$

whence we can conclude that the solution is unique and has the form (5.10). The functions $u_3(t)$ and $\omega_3(t)$ in this case are again described by the last two relations of (5.7), and

$$\varphi(t) \equiv 0(\mathrm{mod}2\pi)$$

Then

$$\tilde{f} = \frac{\tilde{w} - \tilde{\upsilon}}{T}, \quad \tilde{u}(t) = \tilde{\gamma}(t) = \tilde{f}, \quad \tilde{\omega}(t) = \frac{\tilde{w} - \tilde{\upsilon}}{T}t + \tilde{\upsilon}$$

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